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AMTI

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THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

The Association of Mathematics Teachers of India (AMTI) was started in 1965 for the promotion of efforts to improve Mathematics education at all levels. A major aim of the Association is to assist practising teachers of Mathematics in schools in improving their expertise and professional skills. Another important aim is to spot out and foster Mathematical talents in the young. The Association also seeks to disseminate new trends in Mathematics education among parents and public. Other activities of the Association include consultancy services to schools in equipping the Mathematics section of their libraries, in organising children's Mathematics clubs and fairs, in setting up teacher centres in schools, in conducting Mathematics laboratory programmes, in holding practical tests in Mathematics in assisting children in participating investigational projects.

The Association holds "National Mathematical Talent Search Competition" annually and organises Orientation Courses, Seminars and Workshops for teachers and courses for talented students. A national conference is held annually in different parts of the country for teachers to meet and deliberate on important issues of Mathematics education. Innovative teacher award has been instituted to give public recognition to enterprising and pioneering teachers of Mathematics for which entries from teachers are invited.

An award for contributions to the Mathematics Teacher relating to History of Mathematics in the context of mathematics education has been instituted by Prof. R.C. Gupta.

"The Mathematics Teacher (India)" (MT) is the official quarterly journal of the Association and is issued twice a year. It has been approved for use in schools and colleges of education by the Government departments of education in many States. Besides MT the Association also brings out Junior Mathematician (JM), three issues in a year, especially for school students in English and Tamil.

The membership of the Association is open to all those interested in Mathematics and Mathematics Education. The membership fee inclusive of subscription for "The Mathematics Teacher (India)" and effective from April 1993 is as follows:

Subscription for India*

Category	Individual	Institutional
Life	Rs. 1000	Rs. 1500
Annual (Ordinary)	Rs. 100	---
Junior Mathematician – Life	Rs. 500	Rs. 500
Junior Mathematician – Annual	Rs. 50	---

The Journal "The Mathematics Teacher" will be supplied free to all members.
Fifty or more subscriptions to Junior Mathematician will entail 20% discount.

* For countries other than India same figures in US \$.
(inclusive of postage)
i.e instead of rupees read US dollars

EDITORIAL

This issue contains the texts of two of the talks delivered at the 53rd Annual Conference of AMTI conducted between December 26th and 28th December 2019 at Disha Public School, Kota, Rajasthan.

An article on Inequalities is published for those students preparing for Olympiads. In general, inequality problems are posed in almost all olympiads and students often have difficulties in solving them. The current article describes in detail some important inequalities and several applications of them.

Geometry has been neglected to a large extent in the school curriculum. However, many students find geometry an interesting subject. The Iranian Geometry Olympiad, conducted every year, gives an opportunity to students interested in Geometry to showcase their skills in Geometry. This year, India participated in the competition and we are happy to note that several students won medals in the competition. Details are given in the article "Iranian Geometry Olympiad".

CONTENTS

Heuristics of Problem Solving
Dr S.R. Santhanam (AMTI)
(National Awardee for Popularizing Mathematics among masses)
Theme talk – 53rd Annual Conference – Kota

For a common man Mathematics means problem solving. Mathematics books from class 1 to up the ladder contain exercises which in turn contain problem. Mathematics problems are of four types. Right from the easy one to most difficult mathematics poses problems to persons in all walks of life. We come across skilled and seasoned mathematicians to amateurs who take immense interest in solving mathematics problems. Let us see in this paper different modes of solving mathematical problems. Let us restrict ourselves to mathematics of school level.

The following problem was given to different sets of students in high school, higher secondary and college level.

Find integers x which satisfy the cubic equation

$$(x + 1)(x + 2)(x + 3) = 20192019$$

The surprise is a high school boy solved it where as the others could not. Usually geometry problems are difficult to solve because of the deductive logic of the subject. The facts to be remembered to solve a geometry problem is in general much more than any other branch of mathematics. Geometry problems are difficult to solve by school students because of the fact that geometry is not given the right place in school level. This is not only in India but by and large in all countries in the world.

In general, in school level the heuristics of problem solving is not taught. Interdisciplinary approach is an important aspect of problem solving. This aspect is not also concentrated in school level. In this talk the types of problems and their methods of solutions are discussed.

To be an Effective Math teacher

Inder K Rana

Emeritus Fellow, I.I.T. Bombay

President's Address – 53rd Annual Conference – Kota

Why Mathematics?

George Polya (1887 – 1985), a Hungarian mathematician elucidated two aims for school education:

A good and narrow aim, that of turning out employable adults who (eventually) contribute to social and economic development; and a higher aim, that of developing the inner resources of the growing child.

Mathematics helps in achieving both these aims. However, there are issues in mathematics teaching and learning. In National Curriculum Framework 2005, the committee observed that the following are the core areas of concern as far as the mathematics education in India is concerned:

1. A sense of fear and failure regarding mathematics among majority of children

2. A curriculum that disappoints both a talented minority as well as the non-participating majority at the same time.

3. Crude
   methods of assessment that encourage the perception that mathematics as a mechanical computation

4. Lack of teacher preparation and support in the teaching of mathematics.

Role of a teacher

Teaching mathematics is both a challenging and stimulating endeavour.   NCTM   (National   council of Teachers of Mathematics) in its Principles and Standards for School Mathematics mentions "Six Principles for School Mathematics", and one of them on "Teaching" says:

Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well. Students understanding of mathematics, their ability to use it to solve problems, and their confidence in doing mathematics are all shaped by the teaching they encounter in school. To be effective, teachers must understand and be committed to students as learners of mathematics. They must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks. Teachers must be supported with ample opportunities and resources to enhance and refresh their knowledge.

On teaching mathematics, Polya says:

A teacher of mathematics has a great opportunity. If he/she lls his/her allotted time with drilling his students is routine operations, he/she kills their interest, hampers their intellectual development and misuses his/her opportunity. But if he/she challenges the curiosity of his/her students by setting them problems proportional to their knowledge and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of independent thinking.

NCF 2005 also recommends the use of technology:

Technology can greatly aid the process of mathematical exploration, and clever use of such aids can help engage students. Calculators are typically seen as aiding arithmetical operations; while this is true, calculators are of much greater pedagogic value. Indeed, if one asks whether calculators

should be permitted in examinations, the answer is that it is quite unnecessary for examiners to raise questions that necessitate the use of calculators. On the contrary, in a non threatening atmosphere, children can use calculators to study iteration of many algebraic functions. For instance, starting with an arbitrary large number and repeatedly finding the square root to see how soon the sequence converges to 1, is illuminating. Even phenomena like chaos can be easily comprehended with such iterators. If ordinary calculators can offer such possibilities, the potential of graphing calculators and computers for mathematical exploration is far higher.It must be understood that there is a spectrum of technology use in mathematics education, and calculators or computers are at one end of the spectrum. While notebooks and blackboards are the other end, use of graph paper, geo-boards, abacus, geometry boxes etc. is crucial. Innovations in the design and use of such material must be encouraged so that their use makes school mathematics enjoyable and meaningful.

Here are some of the attributes expected of a mathematics teacher:

1. Strong content knowledge

2. Enthusiasm for the subject and the ability to inspire students.

3. Passion for using new teaching techniques and delivering new content

4. Ability to identify students' misconceptions and suggest meaningful solutions.

5. Knowledge and desire to monitor and assess the students progress and identify areas for improvement

6. Integrating activities that promote independent thinking, collaborative learning, writing and talking mathematics skills.

7. Interest in listening to students, openness to suggestions, and the ability to be fair-minded and sensitive to students needs.

8. Designing activities that relate classroom math to the real world.

9. Integrating technological tools in teaching.

10. *Important aspects of problem solving.*

For effective teaching

Teacher education programs, both pre-service and in-service, in India are strongly under the influence of the theory of teaching and learning, called Behaviourism. As a result, teachers remain as transmitters of knowledge. It can be summarized as

1. **Understand – Remember – Reproduce**
   Learning is reacting to external stimuli, either through classical conditioning (reflexive response to stimulus), or through operant conditioning (reinforcement of behavior by reward or punishment).

2. **Understand – Acquire – Analyse – Apply**
   Learning is process of constructing subjective reality based on previous knowledge and objective reality.

Effective lesson planning Teaching a topic/subtopic need to be divided into the following steps:

Step 1: Observe and Explore: Motivating concepts

The aim is to develop the pedagogical and content skill that are necessary for effectiveness as a mathematics teacher. Start your lesson with an activity related to the topic. It can be a video, a puzzle, a game, or a hands on activity. This should motivate initiating Math Talk in the classroom. Asking leading questions, soliciting students responses, and discussions should lead to related to material, the mathematical content, you intend to teach.

Step 2: Define and Prove: Getting to the concepts

This step relates to transacting the main part of the curriculum.Once again, as and when a concept is introduced, there should be an effort to incorporate inquiry and discussion. While proving theorems, ideas of reasoning and proof should be incorporated. A list of frequently asked doubts with explanations be prepared for each topic. Technological tools should be integrated topic wise to strengthen day today teaching.

Step 3: Apply and Evaluate: Using the concepts

This part relates to applying the concepts of a topic/subtopic to problems. It will be a good idea to incorporate a list of non-routine problems that challenge the students thinking. Application of concepts developed to real world problems should also be included. These will also help to evaluate the effectiveness of teaching. **Main aspects of the resource material are the following:**

1. From concrete idea to concepts.

2. Understanding not memorizing.

3. Thinking and doing, not remembering and reproducing.

About 500 teachers from Maharashtra are being trained in this methodology of teaching udder the Rashriya Madhyamik Shiskasha Abhiyaan.

Important Inequalities and Applications
Dr S Muralidharan

Students often find problems involving inequalities difficult. In order to help those preparing for Olympiads, we give some important inequalities and their applications in solving problems.

Rearrangement Inequality

Ten people queue up before a tap to fill their buckets. Each bucket requires a different time to fill. In what order should the people queue up so as to minimize their combined waiting time?" Common sense suggests that they queue up in ascending order of "bucket-filling time". Let us see if our intuition leads us astray. We will denote by $T_1 \leq T_2 \leq \ldots \leq T_{10}$ the times required to fill the respective buckets. If the people queue up in the order suggested, their combined waiting time will be given by $T = 10T_1 + 9T_2 + \cdots + T_{10}$. For a different queueing order, the combined waiting time will be $S = 10S_1 + 9S_2 + \cdots + S_{10}$, where $(S_1, S_2, \ldots, S_{10})$ is a permutation of $(T_1, T_2, \ldots, T_{10})$. The two 10-tuples being different, there is a smallest index i for which $S_i \neq T_i$. Then $S_j = T_i < S_i$ for some $j > i$. Define $S'_i = S_j, S'_j = S_i$ and $S'_k = S_k$ for $k \neq i, j$. Let $S' = 10S'_1 + 9S'_2 + \cdots + S'_{10}$. Then

$$S - S' = (11-i)(S_i - S'_i) + (11-j)(S_j - S'_j) = (S_i - S_j)(j-i) > 0$$

Hence the switching results in a lower combined waiting time. If $(S'_1, S'_2, \ldots, S'_{10}) \neq (T_1, T_2, \ldots, T_{10})$, this switching process can be repeated again. We will reach $(T_1, T_2, \cdots, T_{10})$ in at

most 9 steps. Since the combined waiting time is reduced in each step, T is indeed the minimum combined waiting time. Suppose we have four boxes where the first box contains 2000 Rs notes, second, 500 Rs notes, third, 100 Rs notes and fourth 50 Rs notes. You are allowed to take $3, 5, 7, 10$ notes from these boxes. How will you take the notes to maximize the amount? Which combination results in the minimum amount?

We can generalize this example to the following result.

Theorem 1 *Consider two collections of real numbers in increasing order,*

$$a_1 \leq a_2 \leq a_3 \cdots a_{n-1} \leq a_n \quad and \quad b_1 \leq b_2 \leq b_3 \cdots b_{n-1} \leq b_n$$

For any permutation $(a'_1, a'_2, \ldots, a'_n)$ of $(a_1, a_2, \ldots, a_n)$ we have the following inequalities

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a'_1 b_1 + a'_2 b_2 + \cdots + a'_n b_n \qquad (1)$$
$$a'_1 b_1 + a'_2 b_2 + \cdots + a'_n b_n \geq a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n \qquad (2)$$

Moreover when a_i are all distinct, then equality holds in (1) if and only if

$$(a'_1, a'_2, \ldots, a'_n) = (a_1, a_2, \ldots, a_1) \qquad (3)$$

and in (2) if and only if

$$(a'_1, a'_2, \ldots, a'_n) = (a_n, a_{n-1}, \ldots, a_n) \qquad (4)$$

Proof Let

$$S = a_1 b_1 + a_2 b_2 + \cdots + a_r b_r + \cdots + a_s b_s + a_n b_n \qquad (5)$$
$$S' = a_1 b_1 + a_2 b_2 + \cdots + a_s b_r + \cdots + a_r b_s + a_n b_n \qquad (6)$$

The difference between (5) and (6) is that the coefficients of

b_r and b_s are interchanged, where $r < s$. Now,

$$S - S' = a_r b_r + a_s b_s - a_s b_r - a_r b_s = (b_s - b_r)(a_s - a_r) \geq 0$$

Hence $S \geq S'$. Since any permutation of $(a_1, a_2, \ldots, a_n)$ can be obtained by a sequence of such interchanges, it follows that $S \geq S'$.

To obtain (2), we apply (1) to the numbers

$$-a_n \leq -a_{n-1} \cdots -a_2 \leq -a_1 \quad \text{and} \quad b_1 \leq b_2 \leq b_3 \cdots b_{n-1} \leq b_n$$

to obtain

$$-(a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n) \geq -(a_1' b_1 + a_2' b_2 + \cdots + a_n' b_n)$$

and hence

$$a_1' b_1 + a_2' b_2 + \cdots + a_n' b_n \geq a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n$$

The statement about when equality holds is obvious.

Notation

Sum product $a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$ will be denoted by

$$\left\langle \begin{matrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{matrix} \right\rangle$$

With this notation, we can write

$$\left\langle \begin{matrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{matrix} \right\rangle \geq \left\langle \begin{matrix} a_1' & a_2' & \ldots & a_n' \\ b_1 & b_2 & \ldots & b_n \end{matrix} \right\rangle$$

and

$$\left\langle \begin{matrix} a_n & a_{n-1} & \ldots & a_1 \\ b_1 & b_2 & \ldots & b_n \end{matrix} \right\rangle \leq \left\langle \begin{matrix} a_1' & a_2' & \ldots & a_n' \\ b_1 & b_2 & \ldots & b_n \end{matrix} \right\rangle$$

We deduce the following Corollaries:

Corollary 1 For any permutation $(a_1', a_2', \ldots, a_n')$ of

$(a_1, a_2, \ldots, a_n)$, the following inequality holds:

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq a_1 a_1' + a_2 a_2' + \cdots + a_n a_n'$$

Taking $b_i = a_i$, we have

$$\left\langle \begin{matrix} a_1 & a_1 & \ldots & a_n \\ a_1 & a_2 & \ldots & a_n \end{matrix} \right\rangle \geq \left\langle \begin{matrix} a_1 & a_2 & \ldots & a_n \\ a_1' & a_2' & \ldots & a_n' \end{matrix} \right\rangle$$

Corollary 2 For any permutation $(a_1', a_2', \ldots, a_n')$ of $(a_1, a_2, \ldots, a_n)$, the following inequality holds:

$$\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + \cdots + \frac{a_n'}{a_n} \geq n$$

The sequences $(a_1, a_2, \ldots, a_n)$ and $\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n} \right)$ are oppositely sorted. Hence

$$\left\langle \begin{matrix} a_1 & a_1 & \ldots & a_n \\ \dfrac{1}{a_1} & \dfrac{1}{a_2} & \ldots & \dfrac{1}{a_n} \end{matrix} \right\rangle \leq \left\langle \begin{matrix} a_1' & a_2' & \ldots & a_n' \\ \dfrac{1}{a_1} & \dfrac{1}{a_2} & \ldots & \dfrac{1}{a_n} \end{matrix} \right\rangle$$

Corollary 3 (AM - GM Inequality)

Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Let $g = \sqrt[n]{x_1 x_2 \cdots x_n}$ and consider the sequence

$$a_1 = \frac{x_1}{g}, a_2 = \frac{x_1 x_2}{g^2}, \ldots, a_n = \frac{x_1 x_2 \cdots x_n}{g^n} = 1$$

By Corollary 2,

$$n \leq \frac{a_1}{a_n} + \frac{a_2}{a_1} + \cdots + \frac{a_n}{a_{n-1}} = \frac{x_1}{g} + \frac{x_2}{g} + \cdots + \frac{x_n}{g}$$

and hence $ng \leq x_1 + x_2 + \cdots + x_n$.

Corollary 4 (GM - HM Inequality)

Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Define g and $a_1, a_2, \ldots, a_n$ as in Proof of **Corollary 3**. By **Corollary 2**,

$$n \leq \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1} = \frac{g}{x_1} + \frac{g}{x_2} + \cdots + \frac{g}{x_n}$$

Corollary 5 (Root Mean Square inequality)

Let $x_1, x_2, \ldots, x_n$ be real numbers. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}$$

By **Corollary 1**, we have

$$x_1^2 + x_2^2 + \cdots + x_n^2 = x_1 x_1 + x_2 x_2 + \cdots x_n x_n$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq x_1 x_2 + x_2 x_3 + \cdots x_n x_1$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq x_1 x_3 + x_2 x_4 + \cdots x_n x_2$$

$$\cdots \cdots$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq x_1 x_n + x_2 x_1 + \cdots x_n x_{n-1}$$

Adding, we get

$$n(x_1^2 + x_2^2 + \cdots + x_n^2) \leq (x_1 + x_2 + \cdots + x_n)^2$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Corollary 6 (Cauchy - Schwarz Inequality)

Let $a_1, a_2, \ldots, a_n$ and $b_1, n_2, \ldots, b_n$ be real numbers. Then

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

The result is trivial if $a_1^2 + a_2^2 + \cdots + a_n^2 = 0$ or $b_1^2 + b_2^2 + \cdots + b_n^2 = 0$.

Let

$$A = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}, B = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

Since $A \neq 0$ and $B \neq 0$, we can put

$$x_1 = \frac{a_1}{A}, x_1 = \frac{a_2}{A}, \ldots, x_n = \frac{a_n}{A},$$
$$x_{n+1} = \frac{b_1}{B}, x_{n+2} = \frac{b_2}{B}, \cdots, x_{2n} = \frac{b_n}{B}$$

By **Corollary 1**,

$$\begin{aligned} 2 &= \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{A^2} + \frac{b_1^2 + b_2^2 + \cdots + b_n^2}{B^2} \\ &= x_1^2 + x_2^2 + \cdots x_{2n}^2 \\ &\geq x_1 x_{n+1} + x_2 x_{n+2} + \cdots + x_n x_{2n} \\ &\quad + x_{n+1} x_1 + x_{n+2} x_2 + \cdots + x_{2n} x_n \\ &= \frac{2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)}{AB} \end{aligned}$$

and hence

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

Equality holds if and only if $x_i = x_{n+i} \Leftrightarrow a_iB = b_iA$, that is $\frac{a_i}{b_i}$ is a constant.

Corollary 6 (Chebyshev Inequality)

Let $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$ be any real

numbers. Then

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n \geq \frac{(x_1 + \cdots + x_n)(y_1 + \cdots + y_n)}{n}$$

$$\geq x_1y_n + x_2y_{n-1} + \cdots + x_ny_1$$

Can also be written in the form:

$$\frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{n} \geq \frac{x_1 + \cdots + x_n}{n} \cdot \frac{y_1 + \cdots + y_n}{n}$$

$$\frac{x_1y_n + x_2y_{n-1} + \cdots + x_ny_1}{n} \leq \frac{x_1 + \cdots + x_n}{n} \cdot \frac{y_1 + \cdots + y_n}{n}$$

Problems

1. (Nesbitt Inequality) Let a, b, c be positive real numbers. Prove:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Solution Without loss of generality, assume $a \geq b \geq c$. Then $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$. Thus (a, b, c) and $\left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right)$ are sorted in the same way and hence

$$\left\langle \begin{array}{ccc} a & b & c \\ \frac{1}{b+c} & \frac{1}{c+a} & \frac{1}{a+b} \end{array} \right\rangle \geq \left\langle \begin{array}{ccc} a & b & c \\ \frac{1}{c+a} & \frac{1}{a+b} & \frac{1}{b+c} \end{array} \right\rangle$$

and

$$\left\langle \begin{array}{ccc} a & b & c \\ \frac{1}{b+c} & \frac{1}{c+a} & \frac{1}{a+b} \end{array} \right\rangle \geq \left\langle \begin{array}{ccc} a & b & c \\ \frac{1}{a+b} & \frac{1}{b+c} & \frac{1}{c+a} \end{array} \right\rangle$$

Adding, we get

$$2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \geq 3$$

2. Let a, b, c be positive real numbers. Prove:

$$a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$$

Solution The sequences (a^2, b^2, c^2) and (a, b, c) are sorted in the same way.

$$\left\langle\begin{matrix}a^2 & b^2 & c^2 \\ a & b & c\end{matrix}\right\rangle \geq \left\langle\begin{matrix}a^2 & b^2 & c^2 \\ b & c & a\end{matrix}\right\rangle$$

3. Let a, b, c be positive real numbers. Prove:

$$a^5 + b^5 + c^5 \geq a^4b + b^4c + c^4a$$

Solution The sequences (a^4, b^4, c^4) and (a, b, c) are sorted in the same way.

$$\left\langle\begin{matrix}a^4 & b^4 & c^4 \\ a & b & c\end{matrix}\right\rangle \geq \left\langle\begin{matrix}a^4 & b^4 & c^4 \\ b & c & a\end{matrix}\right\rangle$$

More generally, we have $a^n + b^n + c^n \geq a^{n-1}b + b^{n-1}c + c^{n-1}a$ for positive reals a, b, c and n a positive integer.

4. If a, b, c are positive reals and n a positive integer, show that

$$\frac{a^n}{b+c}+\frac{b^n}{c+a}+\frac{c^n}{a+b} \geq \frac{a^{n-1}+b^{n-1}+c^{n-1}}{2}$$

Solution Consider

$$(a^n, b^n, c^n) \text{ and } \left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right)$$

5. Let a, b, c be positive real numbers such that $abc = 1$. Show that

$$a^3 + b^3 + c^3 + a^3b^3 + b^3c^3 + c^3a^3 \geq 2(a^2b + b^2c + c^2a)$$

Solution The sequences (a^2, b^2, c^2) and (a, b, c) are sorted in the same way.

$$\left\langle \begin{matrix} a^2 & b^2 & c^2 \\ a & b & c \end{matrix} \right\rangle \geq \left\langle \begin{matrix} a^2 & b^2 & c^2 \\ b & c & a \end{matrix} \right\rangle$$

Since $abc = 1$,

$$a^3b^3 + b^3c^3 + c^3a^3 = \frac{1}{c^3} + \frac{1}{a^3} + \frac{1}{b^3}$$

The sequences $(\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2})$ and $(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$ are sorted in the same way.

$$\left\langle \begin{matrix} \frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2} \\ \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \end{matrix} \right\rangle \geq \left\langle \begin{matrix} \frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2} \\ \frac{1}{c}, \frac{1}{a}, \frac{1}{b} \end{matrix} \right\rangle$$

Hence

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq \frac{1}{a^2c} + \frac{1}{b^2a} + \frac{1}{c^2b}$$
$$= b^2c + c^2a + a^2b$$

6. Let a, b, c be positive reals. Show that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{a+b+c}{abc}$$

Solution

$$\left\langle \begin{matrix} \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \\ \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \end{matrix} \right\rangle \geq \left\langle \begin{matrix} \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \\ \frac{1}{c}, \frac{1}{a}, \frac{1}{b} \end{matrix} \right\rangle$$

Hence

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{ac} + \frac{1}{ab} + \frac{1}{bc}$$
$$= \frac{a+b+c}{abc}$$

7. Let a, b, c be positive real numbers. Prove:

$$\frac{b^2 + c^2}{a} + \frac{c^2 + a^2}{b} + \frac{a^2 + b^2}{c} \geq 2(a+b+c)$$

Solution a^2, b^2, c^2 and $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are sorted oppositely.

$$\left\langle \begin{matrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{matrix} \right\rangle \leq \left\langle \begin{matrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ b^2 & c^2 & a^2 \end{matrix} \right\rangle$$

$$\left\langle \begin{matrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a^2 & b^2 & c^2 \end{matrix} \right\rangle \leq \left\langle \begin{matrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ c^2 & a^2 & b^2 \end{matrix} \right\rangle$$

Add to get the desired inequality.

8. Let a, b, c be positive real numbers. Prove:

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$$

Solution Clearing the fractions, we need to prove

$$a^4 c^2 + b^4 a^2 + c^4 b^2 \geq ab^3 c^2 + bc^3 a^2 + ca^3 b^2$$

By Corollary 1 applied to the sequence $(a^2 c, b^2 a, c^2 b)$, we get

$$\left\langle \begin{matrix} a^2 c & b^2 a & c^2 b \\ a^2 c & b^2 a & c^2 b \end{matrix} \right\rangle \geq \left\langle \begin{matrix} a^2 c & b^2 a & c^2 b \\ b^2 a & c^2 b & a^2 c \end{matrix} \right\rangle$$

9. Let a, b, c be positive real numbers. Prove:

$$\frac{a^2 - c^2}{b + c} + \frac{b^2 - a^2}{c + a} + \frac{c^2 - b^2}{a + b} \geq 0$$

Solution We need to prove

$$\frac{a^2}{b + c} + \frac{b^2}{c + a} + \frac{c^2}{a + b} \geq \frac{c^2}{b + c} + \frac{a^2}{c + a} + \frac{b^2}{a + b}$$

Since $\dfrac{1}{b + c}, \dfrac{1}{c + a}, \dfrac{1}{a + b}$ and a^2, b^2, c^2 are sorted in the same way, we have

$$\left\langle \begin{matrix} \frac{1}{b+c} & \frac{1}{c+a} & \frac{1}{a+b} \\ a^2 & b^2 & c^2 \end{matrix} \right\rangle \geq \left\langle \begin{matrix} \frac{1}{b+c} & \frac{1}{c+a} & \frac{1}{a+b} \\ c^2 & a^2 & b^2 \end{matrix} \right\rangle$$

10. Let a, b, c be positive real numbers. Prove:

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$$

Solution We need to prove

$$a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab$$

The sequences (a^2, b^2, c^2) and (bc, ca, ab) are oppositely sorted. Hence

$$\left\langle \begin{matrix} a^2 & b^2 & c^2 \\ bc & ca & ab \end{matrix} \right\rangle \leq \left\langle \begin{matrix} a^2 & b^2 & c^2 \\ a^2 & b^2 & c^2 \end{matrix} \right\rangle$$

11. (IMO 1995) Let a, b, c be positive real numbers such that $abc = 1$. Prove:

$$\frac{1}{a^3(b + c)} + \frac{1}{b^3(c + a)} + \frac{1}{c^3(a + b)} \geq \frac{3}{2}$$

Solution Put $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$. Then we need

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}$$

Since (x, y, z) and $\left(\dfrac{x}{y+z}, \dfrac{y}{z+x}, \dfrac{z}{x+y}\right)$ are sorted in the same order, we get

$$\left\langle \begin{matrix} \frac{x}{x} & \frac{y}{y} & \frac{z}{z} \\ \overline{y+z} & \overline{z+x} & \overline{x+y} \end{matrix} \right\rangle \geq \left\langle \begin{matrix} \frac{x}{y} & \frac{y}{z} & \frac{z}{x} \\ \overline{z+x} & \overline{x+y} & \overline{y+z} \end{matrix} \right\rangle$$

and

$$\left\langle \begin{matrix} \frac{x}{x} & \frac{y}{y} & \frac{z}{z} \\ \overline{y+z} & \overline{z+x} & \overline{x+y} \end{matrix} \right\rangle \geq \left\langle \begin{matrix} \frac{x}{z} & \frac{y}{x} & \frac{z}{y} \\ \overline{x+y} & \overline{y+z} & \overline{x+z} \end{matrix} \right\rangle$$

Adding, we get

$$2\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \right) \geq x+y+z \geq 3(xyz)^{1/3} = 3$$

12. (IMO 1964) a, b, c are lengths of sides of a triangle. Prove that

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc$$

Solution Suppose $a \geq b \geq c$. Then

$$a(b+c-a) - b(c+a-b) = ac - a^2 - bc + b^2$$
$$= c(a-b) - (a-b)(a+b)$$
$$= (a-b)(c-a-b) < 0$$

Thus $a(b+c-a), b(c+a-b), c(a+b-c)$ and a, b, c are

sorted in the reverse order. Hence

$$\left\langle \begin{matrix} a(b+c-a) & b(c+a-b) & c(a+b-c) \\ a & b & c \end{matrix} \right\rangle$$

$$\le \left\langle \begin{matrix} a(b+c-a) & b(c+a-b) & c(a+b-c) \\ b & c & a \end{matrix} \right\rangle$$

and

$$\left\langle \begin{matrix} a(b+c-a) & b(c+a-b) & c(a+b-c) \\ a & b & c \end{matrix} \right\rangle$$

$$\le \left\langle \begin{matrix} a(b+c-a) & b(c+a-b) & c(a+b-c) \\ c & a & b \end{matrix} \right\rangle$$

Adding, the right side simplifies to $6abc$.

13. (IMO 1983) a, b, c are lengths of sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(a-b) \ge 0$$

Solution If s is the semi perimeter of the triangle, we put $x = s - a, y = s - b, z = s - c$. Then x, y, z are positive and $a = y + z, b = z + x, c = x + y$. The given inequality is equivalent to

$$x^3z + y^3x + z^3y \ge x^2yz + y^2zx + z^2xy$$

Since x^2, y^2, z^2 and yz, zx, xy are oppositely sorted, we have

$$\left\langle \begin{matrix} x^2 & y^2 & z^2 \\ yz & zx & xy \end{matrix} \right\rangle \le \left\langle \begin{matrix} x^2 & y^2 & z^2 \\ xz & xy & yz \end{matrix} \right\rangle$$

14. Prove that if a, b, c are the lengths of the sides of a

triangle,

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3$$

Solution Making the substitutions $x = s - a, y = s - b, z = s - c$, where s is the semi perimeter of the triangle, we need to prove

$$\frac{y+z}{2x} + \frac{z+x}{2y} + \frac{x+y}{2z} \geq 3$$

Since (x, y, z) and $\left(\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}\right)$ are oppositely sorted, we have

$$\left\langle \begin{matrix} \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \\ x & y & z \end{matrix} \right\rangle \leq \left\langle \begin{matrix} \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \\ y & z & x \end{matrix} \right\rangle$$

and

$$\left\langle \begin{matrix} \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \\ x & y & z \end{matrix} \right\rangle \leq \left\langle \begin{matrix} \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \\ z & x & y \end{matrix} \right\rangle$$

Hence

$$6 \leq \left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right) + \left(\frac{z}{x} + \frac{x}{y} + \frac{y}{z}\right) = \frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z}$$

15. Let $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$ and $s = a_1 + a_2 + \cdots + a_n$. Show that

$$\frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \cdots + \frac{a_n}{s-a_n} \geq \frac{n}{n-1}$$

Solution The sequences $(a_1, a_2, \ldots, a_n)$ and $\left(\dfrac{1}{s-a_1}, \dfrac{1}{s-a_2}, \ldots, \dfrac{1}{s-a_n}\right)$ are sorted in the same

way. Hence

$$\left\langle \begin{matrix} \frac{1}{s-a_1} & \frac{1}{s-a_2} & \cdots & \frac{1}{s-a_n} \\ a_1 & a_2 & \cdots & a_n \end{matrix} \right\rangle \geq \left\langle \begin{matrix} \frac{1}{s-a_1} & \frac{1}{s-a_2} & \cdots & \frac{1}{s-a_n} \\ a_2 & a_3 & \cdots & a_1 \end{matrix} \right\rangle$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\left\langle \begin{matrix} \frac{1}{s-a_1} & \frac{1}{s-a_2} & \cdots & \frac{1}{s-a_n} \\ a_1 & a_2 & \cdots & a_n \end{matrix} \right\rangle \geq \left\langle \begin{matrix} \frac{1}{s-a_1} & \frac{1}{s-a_2} & \cdots & \frac{1}{s-a_n} \\ a_n & a_1 & \cdots & a_{n-1} \end{matrix} \right\rangle$$

Adding the above, we get

$$n\left(\frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \cdots + \frac{a_n}{s-a_n} \right) \geq n - 1$$

16. (RMO 2017) Let x, y, z be real numbers, each greater than 1. Prove that

$$\frac{x+1}{y+1} + \frac{y+1}{z+1} + \frac{z+1}{x+1} \leq \frac{x-1}{y-1} + \frac{y-1}{z-1} + \frac{z-1}{x-1}$$

Solution We need to prove

$$\frac{x+1}{y+1} - \frac{x-1}{y-1} + \frac{y+1}{z+1} - \frac{y-1}{z-1} + \frac{z+1}{x+1} - \frac{z-1}{x-1} \leq 0$$

$$\Leftrightarrow \frac{2(y-x)}{y^2-1} + \frac{2(z-y)}{z^2-1} + \frac{2(x-z)}{x^2-1} \leq 0$$

$$\Leftrightarrow \frac{x}{x^2-1} + \frac{y}{y^2-1} + \frac{z}{z^2-1} \leq \frac{z}{x^2-1} + \frac{x}{y^2-1} + \frac{y}{z^2-1}$$

Since (x, y, z) and $\left(\dfrac{1}{x^2-1}, \dfrac{1}{y^2-1}, \dfrac{1}{z^2-1} \right)$ are oppositely sorted, we have

$$\left\langle \begin{matrix} \frac{1}{x^2-1} & \frac{1}{y^2-1} & \frac{1}{z^2-1} \\ x & y & z \end{matrix} \right\rangle \leq \left\langle \begin{matrix} \frac{1}{x^2-1} & \frac{1}{y^2-1} & \frac{1}{z^2-1} \\ z & x & y \end{matrix} \right\rangle$$

17. (APMO 1998) Let $a, b, c \in \mathbb{R}^+$. Show that

$$\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) \geq 2\left(1 + \frac{a+b+c}{\sqrt[3]{abc}}\right)$$

Solution Expanding the left hand side and simplifying, the inequality reduces to proving

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{a}{c} + \frac{c}{b} + \frac{b}{a} \geq \frac{2(a+b+c)}{\sqrt[3]{abc}}$$

Put $a = x^3, b = y^3, c = z^3$. We need to prove

$$\frac{x^3}{y^3} + \frac{y^3}{z^3} + \frac{z^3}{x^3} + \frac{x^3}{z^3} + \frac{z^3}{y^3} + \frac{y^3}{x^3} \geq \frac{2(x^3 + y^3 + z^3)}{xyz}$$

By Rearrangement inequality,

$$\left\langle \begin{matrix} \frac{x^2}{y^2} & \frac{y^2}{z^2} & \frac{z^2}{x^2} & \frac{x^2}{z^2} & \frac{z^2}{y^2} & \frac{y^2}{x^2} \\ \frac{x}{y} & \frac{y}{z} & \frac{z}{x} & \frac{x}{z} & \frac{z}{y} & \frac{y}{x} \end{matrix} \right\rangle$$

$$\geq \left\langle \begin{matrix} \frac{x^2}{y^2} & \frac{y^2}{z^2} & \frac{z^2}{x^2} & \frac{x^2}{z^2} & \frac{z^2}{y^2} & \frac{y^2}{x^2} \\ \frac{y}{z} & \frac{z}{x} & \frac{x}{y} & \frac{z}{y} & \frac{y}{x} & \frac{x}{z} \end{matrix} \right\rangle$$

The right hand side simplifies to $\dfrac{2(x^3 + y^3 + z^3)}{xyz}$

18. (IMO 1975) Let $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$ be real numbers. Let $(z_1, z_2, \ldots, z_n)$ be a permutation of $(y_1, y_2, \ldots, y_n)$. Prove that

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2$$
$$\leq (x_1 - z_1)^2 + (x_2 - z_2)^2 + \cdots + (x_n - z_n)^2$$

Solution Note that $\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} z_i^2$. Expanding and

simplifying, the desired inequality is equivalent to

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n \geq x_1z_2 + x_2z_2 + \cdots + x_nz_n$$

which follows from Rearrangement inequality.

19. For $a, b, c > 0$, show that

$$\frac{a^2 + bc}{b + c} + \frac{b^2 + ca}{c + a} + \frac{c^2 + ab}{a + b} \geq a + b + c$$

Solution Since (a^2, b^2, c^2) and $\left(\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}\right)$ are similarly sorted, by Rearrangement inequality, we have

$$\left\langle \begin{array}{ccc} a^2 & b^2 & c^2 \\ \dfrac{1}{b+c} & \dfrac{1}{c+a} & \dfrac{1}{a+b} \end{array} \right\rangle \geq \left\langle \begin{array}{ccc} b^2 & c^2 & a^2 \\ \dfrac{1}{b+c} & \dfrac{1}{c+a} & \dfrac{1}{a+b} \end{array} \right\rangle$$

Thus

$$\frac{a^2 + bc}{b + c} + \frac{b^2 + ca}{c + a} + \frac{c^2 + ab}{a + b}$$
$$\geq \frac{b^2 + bc}{b + c} + \frac{c^2 + ca}{c + a} + \frac{a^2 + ab}{a + b}$$
$$= b + c + a$$

20. $a, b, c > 0$ such that $a^2 + b^2 + c^2 = 1$. Show that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3 + \frac{2(a^3 + b^3 + c^3)}{abc}$$

Solution We need

$$(a^2 + b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq 3 + \frac{2(a^3 + b^3 + c^3)}{abc}$$

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} - \frac{(a_1 + a_2)^2}{x_1 + x_2} = \frac{(a_1 x_2 - a_2 x_1)^2}{x_1 x_2 (x_1 + x_2)} \geq 0$$

Thus

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} \geq \frac{(a_1 + a_2)^2}{x_1 + x_2}$$

Now,

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \frac{a_3^2}{x_3} \geq \frac{(a_1 + a_2)^2}{x_1 + x_2} + \frac{a_3^2}{x_3}$$

$$\geq \frac{(a_1 + a_2 + a_3)^2}{x_1 + x_2 + x_3}$$

and so on.

The *T2 Lemma* is very useful in proving many inequalities. We will illustrate through several examples.

Examples

1. a, b, c, d are positive real numbers such that $a + b + c + d = 1$. Show that

$$\frac{a^2}{a + b} + \frac{b^2}{b + c} + \frac{c^2}{c + d} + \frac{d^2}{d + a} \geq \frac{1}{2}$$

Applying the Lemma, we get

$$\frac{a^2}{a + b} + \frac{b^2}{b + c} + \frac{c^2}{c + d} + \frac{d^2}{d + a} \geq \frac{(a + b + c + d)^2}{2(a + b + c + d)}$$

$$= \frac{a + b + c + d}{2} = \frac{1}{2}$$

2. Let a, b, c be positive real numbers such that $a^2 + b^2 +$

$c^2 = 3abc$. Show that

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \geq \frac{9}{a+b+c}$$

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} = \frac{a^4}{a^3b^2c^2} + \frac{b^4}{b^3c^2a^2} + \frac{c^4}{c^3a^2b^2}$$
$$\geq \frac{(a^2+b^2+c^2)^2}{a^2b^2c^2(a+b+c)}$$
$$= \frac{9a^2b^2c^2}{a^2b^2c^2(a+b+c)}$$
$$= \frac{9}{a+b+c}$$

3. If a, b, c are positive reals, show that

$$\frac{b^2+c^2}{a} + \frac{c^2+a^2}{b} + \frac{a^2+b^2}{c} \geq 2(a+b+c)$$

We have

$$\frac{b^2+c^2}{a} + \frac{c^2+a^2}{b} + \frac{a^2+b^2}{c}$$
$$= \frac{b^2}{a} + \frac{c^2}{a} + \frac{c^2}{b} + \frac{a^2}{b} + \frac{a^2}{c} + \frac{b^2}{c}$$
$$\geq \frac{(b+c+c+a+a+b)^2}{2(a+b+c)}$$
$$= 2(a+b+c)$$

4. If a, b, c, d are positive reals, show that

$$\frac{1}{a+b+c} + \frac{1}{a+b+d} + \frac{1}{a+c+d} + \frac{1}{b+c+d}$$
$$\geq \frac{16}{3(a+b+c+d)}$$

$$\frac{1}{a+b+c} + \frac{1}{a+b+d} + \frac{1}{a+c+d} + \frac{1}{b+c+d}$$

$$\geq \frac{(1+1+1+1)^2}{(a+b+c)+(a+b+d)+(a+c+d)+(b+c+d)}$$

$$= \frac{16}{3(a+b+c+d)}$$

5. Let a, b, c, d be positive real numbers such that $a + b + c + d = 1$. Prove that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}$$

Applying Theorem 3, we get

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{(a+b+c+d)^2}{a+b+b+c+c+d+d+a}$$

$$= \frac{(a+b+c+d)^2}{2(a+b+c+d)}$$

$$= \frac{a+b+c+d}{2} = \frac{1}{2}$$

6. Let a, b, c be positive real numbers with $a^2 + b^2 + c^2 = 3abc$. Prove that

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \geq \frac{9}{a+b+c}$$

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} = \frac{a^2}{ab^2c^2} + \frac{b^2}{bc^2a^2} + \frac{c^2}{ca^2b^2}$$

$$= \frac{\left(\frac{a}{bc}\right)^2}{a} + \frac{\left(\frac{b}{ca}\right)^2}{b} + \frac{\left(\frac{c}{ab}\right)^2}{c}$$

$$\geq \frac{\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right)^2}{a+b+c} \quad \text{by Theorem 3}$$

$$= \frac{\left(\frac{a^2+b^2+c^2}{abc}\right)^2}{a+b+c}$$

$$= \frac{9}{a+b+c} \quad \text{since } a^2 + b^2 + c^2 = 3abc$$

7. For any three positive real numbers a, b, c show that

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a+b+c}{3}$$

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} =$$

$$\frac{a^4}{a(a^2+ab+b^2)} + \frac{b^4}{b(b^2+bc+c^2)} + \frac{c^4}{c(c^2+ca+a^2)}$$

$$\geq \frac{(a^2+b^2+c^2)^2}{a(a^2+ab+b^2) + b(b^2+bc+c^2) + c(c^2+ca+a^2)}$$

$$= \frac{(a^2+b^2+c^2)^2}{(a+b+c)(a^2+b^2+c^2)}$$

$$= \frac{a^2+b^2+c^2}{a+b+c}$$

Now, by Power-Mean inequality,

$$\sqrt{\frac{a^2+b^2+c^2}{3}} \geq \frac{a+b+c}{3}$$

and hence

$$\frac{a^2+b^2+c^2}{a+b+c} \geq \frac{a+b+c}{3}$$

This completes the proof.

8. (*Nesbitt inequality*) Let a, b, c be positive real numbers. Prove:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a^2}{a(b+c)} + \frac{b^2}{b(c+a)} + \frac{c^2}{c(a+b)}$$

$$\geq \frac{(a+b+c)^2}{2(ab+bc+ca)}$$

Hence we need to prove

$$\frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3}{2}$$

$$\frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3}{2} \iff (a+b+c)^2 \geq 3(ab+bc+ca)$$

$$\iff a^2 + b^2 + c^2 \geq ab + bc + ca$$

which clearly holds.

9. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b(b+c)^2} + \frac{b}{c(c+a)^2} + \frac{c}{a(a+b)^2} \geq \frac{9}{4(ab+bc+ca)}$$

We have

$$\frac{a}{b(b+c)^2} + \frac{b}{c(c+a)^2} + \frac{c}{a(a+b)^2}$$

$$= \frac{\frac{a^2}{(b+c)^2}}{ab} + \frac{\frac{b^2}{(c+a)^2}}{bc} + \frac{\frac{c^2}{(a+b)^2}}{ca}$$

$$\geq \frac{\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)^2}{ab+bc+ca}$$

$$\geq \frac{9}{4(ab+bc+ca)} \qquad \text{by Nesbitt inequality}$$

10. Let a, b, c be positive real numbers. Prove:

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a+b+c$$

We have

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} = \frac{a^4}{abc} + \frac{b^4}{bca} + \frac{c^4}{cab}$$

$$\geq \frac{(a^2+b^2+c^2)^2}{3abc}$$

$$= \frac{9\left(\left(\frac{a^2+b^2+c^2}{3}\right)^{\frac{1}{2}}\right)^4}{3abc}$$

$$\geq \frac{9\left(\frac{a+b+c}{3}\right)^4}{3abc}$$

$$= \frac{(a+b+c)^3}{27abc}(a+b+c)$$

$$\geq a+b+c \quad \text{since by AM-GM } \frac{(a+b+c)^3}{27abc} \geq 1$$

11. Let a, b, c be positive real numbers such that $abc = 1$.
    Prove that

$$\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} \geq \frac{3}{2}$$

$$\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} = \frac{a^4}{ab+ac} + \frac{b^4}{bc+ba} + \frac{c^4}{ca+cb}$$

$$\geq \frac{(a^2+b^2+c^2)^2}{2(ab+bc+ca)}$$

$$\geq \frac{(a^2+b^2+c^2)^2}{2(a^2+b^2+c^2)}$$

$$= \frac{a^2+b^2+c^2}{2} \geq \frac{3(abc)^{2/3}}{2}$$

$$= \frac{3}{2}$$

12. (IMO 1995) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

We have

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} = \frac{\frac{1}{a^2}}{a(b+c)} + \frac{\frac{1}{b^2}}{b(c+a)} + \frac{\frac{1}{c^2}}{c(a+b)}$$

$$\geq \frac{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2}{2(ab+bc+ca)}$$

$$= \frac{\left(\frac{ab+bc+ca}{abc}\right)^2}{2(ab+bc+ca)}$$

$$= \frac{ab+bc+ca}{2}$$

$$\geq \frac{3(abc)^{\frac{2}{3}}}{2}$$

$$= \frac{3}{2}$$

Another method Let $a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}$. Then

$xyz = 1$ and

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} = \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}$$

$$\geq \frac{(x+y+z)^2}{2(x+y+z)}$$

$$= \frac{x+y+z}{2} \geq \frac{3\sqrt[3]{xyz}}{2} = \frac{3}{2}$$

13. If a, b, c are positive prove that

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1$$

We have

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} = \frac{a^2}{a(b+2c)} + \frac{b^2}{b(c+2a)} + \frac{c^2}{c(a+2b)}$$

$$\geq \frac{(a+b+c)^2}{3(ab+bc+ca)}$$

$$\geq 1$$

since

$$3(ab+bc+ca) = (ab+bc+ca) + 2(ab+bc+ca)$$

$$\leq a^2 + b^2 + c^2 + 2(ab+bc+ca)$$

$$= (a+b+c)^2$$

14. Prove that for all positive real numbers a, b, c satisfying $a+b+c=1$ the following inequality holds

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \geq \frac{9}{10}$$

We have

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab}$$

$$= \frac{a^2}{a(1+bc)} + \frac{b^2}{b(1+ca)} + \frac{c^2}{c(1+ab)}$$

$$\geq \frac{(a+b+c)^2}{a+b+c+3abc}$$

$$= \frac{1}{1+3abc}$$

Now,

$$3(abc)^{\frac{1}{3}} \leq a+b+c = 1 \implies abc \leq \frac{1}{27}$$

and hence

$$\frac{1}{1+3abc} \geq \frac{9}{10}$$

15. (IMO 1990 Shortlist) Prove that for any four positive real numbers a, b, c, d such that

$$ab + bc + cd + da = 1$$

the following inequality holds:

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}$$

We have

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c}$$

$$= \frac{a^4}{a(b+c+d)} + \frac{b^4}{b(c+d+a)} + \frac{c^4}{c(d+a+b)} + \frac{d^4}{d(a+b+c)}$$

$$\geq \frac{(a^2+b^2+c^2+d^2)^2}{2(ab+ac+ad+bc+bd+cd)}$$

Now,

$$2(ab + ac + ad + bc + bd + cd)$$
$$\leq (a^2 + b^2) + (a^2 + c^2) + (a^2 + d^2)$$
$$+ (b^2 + c^2) + (b^2 + d^2) + (c^2 + d^2)$$
$$= 3(a^2 + b^2 + c^2 + d^2)$$

Also

$$1 = ab + bc + cd + da$$
$$\leq \sqrt{a^2 + b^2 + c^2 + d^2}\sqrt{b^2 + c^2 + d^2 + a^2}$$
$$= a^2 + b^2 + c^2 + d^2$$

Hence

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c}$$
$$\geq \frac{(a^2 + b^2 + c^2 + d^2)^2}{2(ab + ac + ad + bc + bd + cd)}$$
$$\geq \frac{(a^2 + b^2 + c^2 + d^2)^2}{3(a^2 + b^2 + c^2 + d^2)}$$
$$= \frac{a^2 + b^2 + c^2 + d^2}{3}$$
$$\geq \frac{1}{3}$$

16. (IMO 1993 Shortlist) Prove that for any four positive real numbers a, b, c, d

$$\frac{a}{b + 2c + 3d} + \frac{b}{c + 2d + 3a} + \frac{c}{d + 2a + 3b} + \frac{d}{a + 2b + 3c} \geq \frac{2}{3}$$

We have

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c}$$

$$= \frac{a^2}{a(b+2c+3d)} + \frac{b^2}{b(c+2d+3a)} + \frac{c^2}{c(d+2a+3b)}$$

$$+ \frac{d^2}{d(a+2b+3c)}$$

$$\geq \frac{(a+b+c+d)^2}{4(ab+bc+dc+da+ac+db)}$$

Hence we need to prove

$$3(a+b+c+d)^2 \geq 8(ab+bc+cd+da+ac+db) \quad (7)$$

Since

$$(a+b+c+d)^2 = (a^2+b^2+c^2+d^2)+2(ab+bc+cd+da+ac+bd)$$

establishing (7) is equivalent to proving

$$3(a+b+c+d)^2 \geq 4(a+b+c+d)^2 - 4(a^2+b^2+c^2+d^2)$$

or equivalently, proving

$$(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$$

This follows from Cauchy-Schwarz:

$$1 \cdot a + 1 \cdot b + 1 \cdot c + 1 \cdot d \leq \sqrt{1^2+1^2+1^2+1^2}\sqrt{a^2+b^2+c^2+d^2}$$

17. For a, b, c positive reals, show that

$$\frac{a^2+1}{b+c} + \frac{b^2+1}{c+a} + \frac{c^2+1}{a+b} \geq 3$$

We have

$$\frac{a^2+1}{b+c} + \frac{b^2+1}{c+a} + \frac{c^2+1}{a+b}$$

$$= \left(\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}\right)$$

$$+ \left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right)$$

$$\geq \frac{(a+b+c)^2}{2(a+b+c)} + \frac{(1+1+1)^2}{2(a+b+c)}$$

$$= \frac{a+b+c}{2} + \frac{9}{2(a+b+c)}$$

$$\geq 2\sqrt{\frac{a+b+c}{2} \times \frac{9}{2(a+b+c)}}$$

$$= 3$$

18. For a, b, c positive reals, prove that

$$\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} \geq a+b+c$$

We have

$$\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a}$$

$$= \left(\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a}\right)$$

$$+ \left(\frac{b^2}{a+b} + \frac{c^2}{b+c} + \frac{a^2}{c+a}\right)$$

$$\geq \frac{(a+b+c)^2}{2(a+b+c)} + \frac{(b+c+a)^2}{2(a+b+c)}$$

$$= a+b+c$$

19. For a, b, c positive reals, prove that

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq ab + bc + ca$$

We have

$$
\begin{aligned}
\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} &= \frac{a^4}{ab} + \frac{b^4}{bc} + \frac{c^4}{ca} \\
&\geq \frac{(a^2 + b^2 + c^2)^2}{ab + bc + ca} \\
&\geq \frac{(ab + bc + ca)^2}{ab + bc + ca} \\
&\qquad \text{since } a^2 + b^2 + c^2 \geq ab + bc + ca \\
&= ab + bc + ca
\end{aligned}
$$

20. For a, b, c positive reals, prove that

$$a^3b + b^3c + c^3a \geq abc(a + b + c)$$

We have

$$
\begin{aligned}
a^3b + b^3c + c^3a &= \frac{a^2}{\frac{1}{ab}} + \frac{b^2}{\frac{1}{bc}} + \frac{c^2}{\frac{1}{ca}} \\
&\geq \frac{(a + b + c)^2}{\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}} \\
&= \frac{(a + b + c)^2}{\frac{a+b+c}{abc}} \\
&= abc(a + b + c)
\end{aligned}
$$

21. For a, b, c positive reals such that $a^2 + b^2 + c^2 = 3$, prove that

$$\frac{a^3}{b + 2c} + \frac{b^3}{c + 2a} + \frac{c^3}{a + 2b} \geq 1$$

We have

$$\frac{a^3}{b+2c} + \frac{b^3}{c+2a} + \frac{c^3}{a+2b}$$

$$= \frac{a^4}{a(b+2c)} + \frac{b^4}{b(c+2a)} + \frac{c^4}{c(a+2b)}$$

$$\geq \frac{(a^2+b^2+c^2)^2}{3(ab+bc+ca)}$$

$$= \frac{9}{3(ab+bc+ca)}$$

Now,

$$3(ab+bc+ca) \leq 3(a^2+b^2+c^2) = 9$$

and hence

$$\frac{9}{3(ab+bc+ca)} \geq 1$$

and the desired inequality follows.

22. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(b+c)(b+a)} + \frac{c^2}{(c+a)(c+b)} \geq \frac{3}{4}$$

We have

$$\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(b+c)(b+a)} + \frac{c^2}{(c+a)(c+b)}$$

$$\geq \frac{(a+b+c)^2}{a^2+b^2+c^2+3(ab+bc+ca)}$$

Hence we need to prove

$$\frac{(a+b+c)^2}{a^2+b^2+c^2+3(ab+bc+ca)} \geq \frac{3}{4}$$

$$\iff 4(a+b+c)^2 \geq 3(a^2+b^2+c^2) + 9(ab+bc+ca)$$

$$\iff a^2+b^2+c^2 \geq ab+bc+ca$$

which obviously holds (by Cauchy-Schwarz).

23. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a^2}{2+b+c^2} + \frac{b^2}{2+c+a^2} + \frac{c^2}{2+a+b^2} \geq \frac{(a+b+c)^2}{12}$$

We have

$$\frac{a^2}{2+b+c^2} + \frac{b^2}{2+c+a^2} + \frac{c^2}{2+a+b^2} \geq \frac{(a+b+c)^2}{9+a+b+c}$$

Now,

$$\frac{a+b+c}{3} \leq \left(\frac{a^2+b^2+c^2}{3}\right)^{\frac{1}{2}} = 1$$

This can also be seen by applying Cauchy-Schwarz:

$$(a+b+c)^2 = (a\cdot 1 + b\cdot 1 + c\cdot 1)^2 \leq (a^2+b^2+c^2)(1^2+1^2+1^2) = 9$$

Hence $9+a+b+c \leq 12$ and the desired inequality follows.

24. Let a, b, c, d be positive real numbers such that $a + b + c + d = 1$. Prove that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}$$

We have

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{(a+b+c+d)^2}{2(a+b+c+d)}$$
$$= \frac{1}{2}$$

25. Let a, b, c, d, e be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+e} + \frac{d}{e+a} + \frac{e}{a+b} \geq \frac{5}{2}$$

We have

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+e} + \frac{d}{e+a} + \frac{e}{a+b}$$

$$= \frac{a^2}{a(b+c)} + \frac{b^2}{b(c+d)} + \frac{c^2}{c(d+e)} + \frac{d^2}{d(e+a)} + \frac{e^2}{e(a+b)}$$

$$\geq \frac{(a+b+c+d+e)^2}{a(b+c) + b(c+d) + c(d+e) + d(e+a) + e(a+b)}$$

Hence we need to prove

$$\frac{(a+b+c+d+e)^2}{a(b+c) + b(c+d) + c(d+e) + d(e+a) + e(a+b)} \geq \frac{5}{2}$$

$$\iff 2(a+b+c+d+e)^2$$

$$\geq 5(ab + ac + ad + ae + bc + bd + be + cd + ce + de)$$

$$\iff 2(a^2 + b^2 + c^2 + d^2 + e^2)$$

$$\geq ab + ac + ad + ae + bc + bd + be + cd + ce + de$$

$$\iff 2(a^2 + b^2 + c^2 + d^2 + e^2)$$

$$\geq (ab + bc + cd + de + ea) + (ac + bd + ce + da + eb)$$

Now by Cauchy-Schwarz, we have

$$ab + bc + cd + de + ea \leq \sqrt{a^2 + b^2 + c^2 + d^2 + e^2}$$

$$\times \sqrt{b^2 + c^2 + d^2 + e^2 + a^2}$$

$$= a^2 + b^2 + c^2 + d^2 + e^2$$

$$ac + bd + ce + da + eb \leq \sqrt{a^2 + b^2 + c^2 + d^2 + e^2}$$

$$\times \sqrt{c^2 + d^2 + e^2 + a^2 + b^2}$$

$$= a^2 + b^2 + c^2 + d^2 + e^2$$

Another way to see this is as follows:

Let $S = a + b + c + d + e$.

$$\frac{(a+b+c+d+e)^2}{a(b+c)+b(c+d)+c(d+e)+d(e+a)+e(a+b)}$$

$$= \frac{(a+b+c+d+e)^2}{\frac{1}{2}(a(S-a)+b(S-b)+c(S-c)+d(S-d)+e(S-e))}$$

$$= \frac{(a+b+c+d+e)^2}{\frac{1}{2}\left((a+b+c+d+e)^2 - (a^2+b^2+c^2+d^2+e^2)\right)}$$

Hence we need to prove

$$(a+b+c+d+e)^2 \leq 5(a^2+b^2+c^2+d^2+e^2)$$

which follows from Cauchy-Schwarz inequality.

26. Let a, b, c, d be positive real numbers. Prove that

$$\frac{a}{a+2b+c} + \frac{b}{b+2c+d} + \frac{c}{c+2d+a} + \frac{d}{d+2a+b} \geq 1$$

We have

$$\frac{a}{a+2b+c} + \frac{b}{b+2c+d} + \frac{c}{c+2d+a} + \frac{d}{d+2a+b}$$

$$= \frac{a^2}{a(a+2b+c)} + \frac{b^2}{b(b+2c+d)} + \frac{c^2}{c(c+2d+a)}$$

$$\quad + \frac{d^2}{d(d+2a+b)}$$

$$\geq \frac{(a+b+c+d)^2}{a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)}$$

$$= 1$$

27. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$$

Prove that

$$\frac{a_1^2}{a_1 + b_1} + \frac{a_2^2}{a_2 + b_2} + \cdots + \frac{a_n^2}{a_n + b_n} \geq \frac{1}{2}(a_1 + a_2 + \cdots + a_n)$$

We have

$$\frac{a_1^2}{a_1 + b_1} + \frac{a_2^2}{a_2 + b_2} + \cdots + \frac{a_n^2}{a_n + b_n}$$

$$\geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{(a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n)}$$

$$= \frac{(a_1 + a_2 + \cdots + a_n)^2}{2(a_1 + a_2 + \cdots + a_n))}$$

$$= \frac{1}{2}(a_1 + a_2 + \cdots + a_n)$$

28. Let a, b, c are the lengths of sides of a triangle. Prove
that
$$\frac{a}{b + c - a} + \frac{b}{c + a - b} + \frac{c}{a + b - c} \geq 3$$

Since $a + b > c$, $b + c > a$, $a + b > c$, the denominators
are all positive. We have

$$\frac{a}{b + c - a} + \frac{b}{c + a - b} + \frac{c}{a + b - c}$$

$$= \frac{a^2}{a(b + c - a)} + \frac{b^2}{b(c + a - b)} + \frac{c^2}{c(a + b - c)}$$

$$\geq \frac{(a + b + c)^2}{2(ab + bc + ca) - (a^2 + b^2 + c^2)}$$

We need to prove

$$\frac{(a + b + c)^2}{2(ab + bc + ca) - (a^2 + b^2 + c^2)} \geq 3$$

$$\Longleftrightarrow (a + b + c)^2 \geq 6(ab + bc + ca) - 3(a^2 + b^2 + c^2)$$

$$\Longleftrightarrow 4(a^2 + b^2 + c^2) \geq 4(ab + bc + ca)$$

which holds.

29. Let a, b, c are the lengths of sides of a triangle. Prove that

$$\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a} \geq 3$$

We have

$$\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a}$$

$$= \frac{a^2}{a(c+a-b)} + \frac{b^2}{b(a+b-c)} + \frac{c^2}{c(b+c-a)}$$

$$\geq \frac{a+b+c)^2}{a^2+b^2+c^2}$$

Hence we need to prove

$$\frac{a+b+c)^2}{a^2+b^2+c^2} \geq 3$$

or equivalently,

$$ab + bc + ca \geq a^2 + b^2 + c^2$$

which is clearly false! Hence direct application of *T2 Lemma* does not work.

If $s = \frac{a+b+c}{2}$ is the semi-perimeter of the triangle, let us put

$$x = s - a = \frac{b+c-a}{2}, \quad y = s - b = \frac{c+a-b}{2},$$

$$z = s - c = \frac{a+b-c}{2}$$

Hence x, y, z are positive and

$$a = y + z, \quad b = z + x, \quad c = x + y$$

The inequality to be proved is now

$$\frac{y+z}{2y} + \frac{z+x}{2z} + \frac{x+y}{2x} \ge 3$$

Now we can write

$$\frac{y+z}{2y} + \frac{z+x}{2z} + \frac{x+y}{2x} = \frac{3}{2} + \frac{1}{2}\left(\frac{z}{y} + \frac{x}{z} + \frac{y}{x}\right)$$

$$\ge \frac{3}{2} + \frac{1}{2} \cdot 3 \cdot \left(\frac{z}{y} \cdot \frac{x}{z} \cdot \frac{y}{x}\right)^{\frac{1}{3}}$$

$$= 3$$

30. If a, b, c are sides of a triangle and $s = \frac{a+b+c}{2}$ is the semi-perimeter, prove that

$$\frac{(s-a)^2}{bc} + \frac{(s-b)^2}{ca} + \frac{(s-c)^2}{ab} \ge \frac{3}{4}$$

We have

$$\frac{(s-a)^2}{bc} + \frac{(s-b)^2}{ca} + \frac{(s-c)^2}{ab}$$

$$\ge \frac{((s-a)+(s-b)+(s-c))^2}{bc+ca+ab}$$

$$= \frac{\left(\frac{a+b+c}{2}\right)^2}{bc+ca+ab}$$

Hence we need to prove

$$(a+b+c)^2 \ge 3(bc+ca+ab)$$

or equivalently,

$$a^2 + b^2 + c^2 \ge bc + ca + ab$$

which clearly holds.

31. For positive real numbers a, b, c such that $abc = 1$, prove that
$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} \geq 1$$

We have
$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1}$$
$$= \frac{a^2}{a(b+c+1)} + \frac{b^2}{b(c+a+1)} + \frac{c^2}{(a+b+1)}$$
$$\geq \frac{(a+b+c)^2}{(a+b+c) + 2(ab+bc+ca)}$$

Hence we need to prove
$$\frac{(a+b+c)^2}{(a+b+c) + 2(ab+bc+ca)} \geq 1$$
$$\Longleftrightarrow (a+b+c)^2 \geq (a+b+c) + 2(ab+bc+ca)$$
$$\Longleftrightarrow a^2 + b^2 + c^2 \geq a+b+c$$

Now, by Power-Mean inequality, we have
$$\frac{a^2 + b^2 + c^2}{3} \geq \left(\frac{a+b+c}{3}\right)^2$$
$$\Longrightarrow a^2 + b^2 + c^2 \geq \frac{a+b+c}{3} \times (a+b+c)$$
$$\Longrightarrow a^2 + b^2 + c^2 \geq \frac{a+b+c}{3} \times 3(abc)^{\frac{1}{3}},$$
$$\text{by AM-GM inequality}$$
$$= a+b+c \quad \text{since } abc = 1$$

32. a, b, c are real numbers such that $ab + bc + ca \geq 0$. Show that
$$\frac{ab}{a^2 + b^2} + \frac{bc}{b^2 + c^2} + \frac{ca}{c^2 + a^2} \geq -\frac{1}{2}$$
Here, applying the Theorem 3 directly is cumbersome.

Instead, we first add $\frac{1}{2}$ to each fraction on the left. We need to prove

$$\frac{ab}{a^2+b^2} + \frac{1}{2} + \frac{bc}{b^2+c^2} + \frac{1}{2} + \frac{ca}{c^2+a^2} + \frac{1}{2} \geq 1$$

Now, the left hand side can be written as

$$\frac{(a+b)^2}{2(a^2+b^2)} + \frac{(b+c)^2}{2(b^2+c^2)} + \frac{(c+a)^2}{2(c^2+a^2)}$$

$$\geq \frac{(2a+2b+2c)^2}{4(a^2+b^2+c^2)}$$

$$= \frac{(a+b+c)^2}{a^2+b^2+c^2}$$

$$= 1 + \frac{2(ab+bc+ca)}{a^2+b^2+c^2}$$

$$\geq 1 \qquad \text{since } ab+bc+ca \geq 0$$

33. If a, b, c, d are positive real numbers such that $a^2 + b^2 + c^2 + d^2 = 1$, show that

$$\frac{a^5}{b+c+d} + \frac{b^5}{a+c+d} + \frac{c^5}{a+b+d} + \frac{d^5}{a+b+c} \geq \frac{1}{12}$$

We have

$$\frac{a^5}{b+c+d} + \frac{b^5}{a+c+d} + \frac{c^5}{a+b+d} + \frac{d^5}{a+b+c}$$

$$= \frac{a^6}{a(b+c+d)} + \frac{b^6}{b(a+c+d)} + \frac{c^6}{c(a+b+d)} + \frac{d^6}{d(a+b+c)}$$

$$\geq \frac{(a^3+b^3+c^3+d^3)^2}{2(ab+ac+ad+bc+bd+cd+da)}$$

Now,

$$2(ab + ac + ad + bc + bd + cd + da)$$
$$= (a + b + c + d)^2 - (a^2 + b^2 + c^2 + d^2)$$
$$= (a + b + c + d)^2 - 1$$

Hence we need to prove

$$\frac{(a^3 + b^3 + c^3 + d^3)^2}{(a + b + c + d)^2 - 1} \geq \frac{1}{12}$$

Now, by Power-Mean inequality,

$$\left(\frac{a^3 + b^3 + c^3 + d^3}{4} \right)^{\frac{1}{3}} \geq \left(\frac{a^2 + b^2 + c^2 + d^2}{4} \right)^{\frac{1}{2}}$$
$$\geq \frac{a + b + c + d}{4}$$

Hence

$$a^3 + b^3 + c^3 + d^3 \geq \frac{1}{2}$$
$$a + b + c + d \leq 2$$

Hence

$$\frac{(a^3 + b^3 + c^3 + d^3)^2}{(a + b + c + d)^2 - 1} \geq \frac{\frac{1}{4}}{2^2 - 1} = \frac{1}{12}$$

More generally, if $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1^2 + a_2^2 + \cdots + a_n^2 = 1$$

then

$$\frac{a_1^5}{a_2 + a_3 + \cdots + a_n} + \frac{a_2^5}{a_1 + a_3 + \cdots + a_n} + \cdots$$

$$+ \frac{a_n^5}{a_1 + a_2 + \cdots + a_{n-1}}$$

$$\geq \frac{1}{n(n-1)}$$

34. Prove that for any positive real numbers a, b, c,

$$\frac{a^2 + bc}{b + c} + \frac{b^2 + ca}{c + a} + \frac{c^2 + ab}{a + b} \geq a + b + c$$

We have

$$\frac{a^2 + bc}{b + c} + \frac{b^2 + ca}{c + a} + \frac{c^2 + ab}{a + b}$$

$$= \frac{(a^2 + bc)^2}{(a^2 + bc)(b + c)} + \frac{(b^2 + ca)^2}{(b^2 + ca)(c + a)} + \frac{(c^2 + ab)^2}{(c^2 + ab)(a + b)}$$

$$\geq \frac{(a^2 + b^2 + c^2 + ab + bc + ca)^2}{2\{ab(a + b) + bc(b + c) + ca(c + a)\}}$$

Hence we need to prove

$$(a^2 + b^2 + c^2 + ab + bc + ca)^2$$

$$\geq 2(a + b + c)\{ab(a + b) + bc(b + c) + ca(c + a)\}$$

Expanding and canceling equal terms, the above reduces to proving

$$a^4 + b^4 + c^4 \geq a^2 b^2 + b^2 c^2 + c^2 a^2$$

which obviously holds.

Remark Consider the function

$$f(a, b, c) = (a^2 + b^2 + c^2 + ab + bc + ca)^2 -$$
$$2(a + b + c) \{ab(a + b) + bc(b + c) + ca(c + a)\}$$

We need to show that

$$f(a, b, c) = a^4 + b^4 + c^4 - (a^2 b^2 + b^2 c^2 + c^2 a^2)$$

Of course, we can do this by brute-force but the following argument does it elegantly.

f is a homogeneous and symmetric function of degree 4 in a, b, c. Hence the only terms that can occur in the expansion of f are $a^i b^j c^j$ where $i + j + k = 4$. The coefficient of a^4 is clearly 1 since a^4 term occurs only in the expansion of the first term

$$(a^2 + b^2 + c^2 + ab + bc + ca)^2$$

Due to symmetry of f, the coefficients of b^4 and c^4 are also equal to 1.

Now, ab^3 can be obtained from $f(a, b, 0)$. We have

$$f(a, b, 0) = (a^2 + b^2 + ab)^2 - 2(a + b)ab(a + b)$$
$$= a^4 + b^4 + 3a^2 b^2 - 4a^2 b^2 = a^4 + b^4 - a^2 b^2$$

Thus the term ab^3 can not occur in the expansion of $f(a, b, c)$. By symmetry, none of the terms $a^3 b, bc^3, b^3 c, ca^3, c^3 a$ occur in $f(a, b, c)$.

The term $a^2 b^2$ has coefficient 1 when we expand the first term and has coefficient 2 when we expand the second term (from multiplying $2a$ and $ab(a + b)$). Thus the

coefficient of a^2b^2 is -1. By symmetry, the coefficients of b^2c^2 and c^2a^2 are also -1.

Also, the coefficient of abc^2 in $(a^2+b^2+c^2+ab+bc+ca)^2$ is 4 and in $2(a+b+c)\{ab(a+b)+bc(b+c)+ca(c+a)\}$ is also 4 and hence the terms cancel out. Similarly none of the terms ab^2c, a^2cb occur in the expansion of $f(a,b,c)$. Thus,

$$f(a,b,c) = a^4 + b^4 + c^4 - (a^2b^2 + b^2c^2 + c^2a^2)$$

35. (Balkan 2002) Prove that

$$\frac{2}{b(a+b)} + \frac{2}{c(b+c)} + \frac{2}{a(c+a)} \geq \frac{27}{(a+b+c)^2}$$

for positive real numbers a, b, c. We have

$$\frac{2}{b(a+b)} + \frac{2}{c(b+c)} + \frac{2}{a(c+a)}$$

$$= 2\left(\frac{\frac{1}{b}}{a+b} + \frac{\frac{1}{c}}{b+c} + \frac{\frac{1}{a}}{c+a}\right)$$

$$\geq 2\frac{\left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}} + \frac{1}{\sqrt{a}}\right)^2}{2(a+b+c)}$$

$$= \frac{\left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}} + \frac{1}{\sqrt{a}}\right)^2}{(a+b+c)}$$

Hence we need to prove

$$(a+b+c)\left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}} + \frac{1}{\sqrt{a}}\right)^2 \geq 27$$

Now, by AM-GM inequality, we have,

$$a + b + c \geq 3(abc)^{\frac{1}{3}}$$

$$\left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}} + \frac{1}{\sqrt{a}}\right)^2 \geq \frac{9}{(abc)^{\frac{1}{3}}}$$

Multiplying the above two inequalities, we obtain the desired result.

36. (Estonia 2004) Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{1}{1 + 2ab} + \frac{1}{1 + 2bc} + \frac{1}{1 + 2ca} \geq 1$$

Since $2ab \leq a^2 + b^2$, $2bc \leq b^2 + c^2$ and $2ca \leq c^2 + a^2$, we have

$$\frac{1}{1 + 2ab} + \frac{1}{1 + 2bc} + \frac{1}{1 + 2ca}$$
$$\geq \frac{1}{1 + a^2 + b^2} + \frac{1}{1 + b^2 + c^2} + \frac{1}{1 + c^2 + a^2}$$
$$\geq \frac{(1 + 1 + 1)^2}{3 + 2(a^2 + b^2 + c^2)} = 1$$

37. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, show that

$$\frac{1}{1 + ab} + \frac{1}{1 + bc} + \frac{1}{1 + ca} \geq \frac{3}{2}$$

We have

$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca}$$

$$\geq \frac{(1+1+1)^2}{3 + ab + bc + ca} = \frac{9}{3 + ab + bc + ca}$$

$$\geq \frac{9}{6} \quad \text{since } ab + bc + ca \leq a^2 + b^2 + c^2 = 3$$

$$= \frac{3}{2}$$

38. (Greece 2007) If a, b, c are the sides of a triangle, show that

$$\frac{(a+b-c)^4}{b(b+c-a)} + \frac{(b+c-a)^4}{c(c+a-b)} + \frac{(c+a-b)^4}{a(a+b-c)} \geq ab + bc + ca$$

We use the substitution $a = y + z, b = z + x, c = x + y$ where $x, y, z > 0$. The given inequality transforms to

$$\frac{(2z)^4}{2x(x+z)} + \frac{(2x)^4}{2y(y+x)} + \frac{(2y)^4}{2z(y+z)}$$
$$\geq (x+y)(y+z) + (y+z)(z+x) + (z+x)(x+y)$$

Now,

$$\frac{(2z)^4}{2x(x+z)} + \frac{(2x)^4}{2y(y+x)} + \frac{(2y)^4}{2z(y+z)}$$

$$= 8 \left(\frac{z^4}{x(x+z)} + \frac{x^4}{y(y+x)} + \frac{y^4}{z(y+z)} \right)$$

$$\geq 8 \left(\frac{(x^2+y^2+z^2)^2}{x^2+y^2+z^2+xy+yz+zx} \right)$$

$$\geq 8 \left(\frac{(x^2+y^2+z^2)^2}{2(x^2+y^2+z^2)} \right)$$

$$\text{since } x^2 + y^2 + z^2 \geq xy + yz + zx$$

$$= 4(x^2+y^2+z^2)$$

Also,

$$(x+y)(y+z)+(y+z)(z+x)+(z+x)(x+y)$$
$$= (x^2+y^2+z^2)+3(xy+yz+zx)$$
$$\leq 4(x^2+y^2+z^2)$$

This completes the proof.

39. If a, b, c are positive real numbers such that $a+b+c=1$, prove that

$$\frac{(b+c)^5}{a} + \frac{(c+a)^5}{b} + \frac{(a+b)^5}{c} \geq \frac{32}{9}(ab+bc+ca)$$

We have

$$\frac{(b+c)^5}{a} + \frac{(c+a)^5}{b} + \frac{(a+b)^5}{c}$$
$$= \frac{(b+c)^6}{a(b+c)} + \frac{(c+a)^6}{b(c+a)} + \frac{(a+b)^6}{c(a+b)}$$
$$\geq \frac{\left((b+c)^3 + (c+a)^3 + (a+b)^3\right)^2}{a(b+c)+b(c+a)+c(a+b)}$$
$$= \frac{\left((b+c)^3 + (c+a)^3 + (a+b)^3\right)^2}{2(ab+bc+ca)}$$

Hence we need to prove

$$(b+c)^3 + (c+a)^3 + (a+b)^3 \geq \frac{8}{3}(ab+bc+ca)$$

By Power-Means inequality,

$$\left(\frac{(b+c)^3 + (c+a)^3 + (a+b)^3}{3}\right)^{\frac{1}{3}}$$
$$\geq \frac{(b+c) + (c+a) + (a+b)}{3} = \frac{2}{3}$$

Hence
$$(b + c)^3 + (c + a)^3 + (a + b)^3 \geq \frac{8}{9}$$

Also,

$$3(ab+bc+ca) \leq (a^2+b^2+c^2)+2(ab+bc+ca) = (a+b+c)^2 = 1$$

and hence
$$ab + bc + ca \leq \frac{1}{3}$$

Thus

$$(b+c)^3+(c+a)^3+(a+b)^3 \geq \frac{8}{9} = \frac{8}{3} \times \frac{1}{3} \geq \frac{8}{3}(ab+bc+ca)$$

This completes the proof.

40. If a, b, c are positive real numbers such that $a^3+b^3+c^3 = 1$, prove that

$$\frac{1}{a^5(b^2 + c^2)^2} + \frac{1}{b^5(c^2 + a^2)^2} + \frac{1}{c^5(a^2 + b^2)^2} \geq \frac{81}{4}$$

We have

$$\frac{1}{a^5(b^2 + c^2)^2} + \frac{1}{b^5(c^2 + a^2)^2} + \frac{1}{c^5(a^2 + b^2)^2}$$

$$= \frac{\frac{1}{a^2(b^2+c^2)^2}}{a^3} + \frac{\frac{1}{b^2(c^2+a^2)^2}}{b^3} + \frac{\frac{1}{c^2(a^2+b^2)^2}}{c^3}$$

$$\geq \frac{\left(\frac{1}{a(b^2+c^2)} + \frac{1}{b(c^2+a^2)} + \frac{1}{c(a^2+b^2)}\right)^2}{a^3 + b^3 + c^3}$$

$$= \left(\frac{1}{a(b^2 + c^2)} + \frac{1}{b(c^2 + a^2)} + \frac{1}{c(a^2 + b^2)}\right)^2$$

Hence we need to prove that

$$\frac{1}{a(b^2 + c^2)} + \frac{1}{b(c^2 + a^2)} + \frac{1}{c(a^2 + b^2)} \geq \frac{9}{2}$$

By A.M-G.M inequality,

$$\frac{3}{\frac{1}{a(b^2+c^2)} + \frac{1}{b(c^2+a^2)} + \frac{1}{c(a^2+b^2)}} \leq \frac{a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2)}{3}$$

and hence

$$\frac{1}{a(b^2+c^2)} + \frac{1}{b(c^2+a^2)} + \frac{1}{c(a^2+b^2)}$$
$$\geq \frac{9}{a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2)}$$

Thus it is sufficient to establish that

$$a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2) \leq 2$$

We will show that

$$a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2) \leq 2(a^3+b^3+c^3)$$

and since $a^3 + b^3 + c^3 = 1$, this will complete the proof.

$$a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2)$$
$$= ab(a+b) + bc(b+c) + ca(c+a)$$
$$\leq \frac{1}{2}\left((a^2+b^2)(a+b) + (b^2+c^2)(b+c) + (c^2+a^2)(c+a)\right)$$
$$= \frac{1}{2}\left(2(a^3+b^3+c^3) + (a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2))\right)$$
$$= (a^3+b^3+c^3) + \frac{1}{2}\left(a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2)\right)$$

Thus

$$\frac{1}{2}\left(a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2)\right) \leq a^3+b^3+c^3$$

or,

$$a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) \leq 2(a^3 + b^3 + c^3)$$

41. If a, b, c are positive real numbers such that $a^3 + b^3 + c^3 = 1$, prove that

$$\sum \frac{a}{bc(b+c)} \geq \frac{27}{2(a+b+c)^2}$$

We have

$$\sum \frac{a}{bc(b+c)} = \sum \frac{a^2}{abc(b+c)}$$
$$\geq \frac{(a+b+c)^2}{2abc(a+b+c)}$$
$$= \frac{a+b+c}{2abc}$$
$$= \frac{(a+b+c)^3}{2abc(a+b+c)^2}$$
$$\geq \frac{27}{2(a+b+c)^2}$$
$$\text{since } \frac{(a+b+c)^3}{abc} \geq 27,$$
$$\text{by A.M-G.M inequality}$$

42. If a, b, c are positive real numbers, show that

$$\frac{a^3}{a+2b+3c} + \frac{b^3}{b+2c+3a} + \frac{c^3}{c+2a+3b} \geq \frac{a^2+b^2+c^2}{6}$$

We have

$$\frac{a^3}{a+2b+3c} + \frac{b^3}{b+2c+3a} + \frac{c^3}{c+2a+3b}$$

$$= \frac{a^4}{a(a+2b+3c)} + \frac{b^4}{b(b+2c+3a)} + \frac{c^4}{c(c+2a+3b)}$$

$$\geq \frac{(a^2+b^2+c^2)^2}{a(a+2b+3c)+b(b+2c+3a)+c(c+2a+3b)}$$

$$= \frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+5(ab+bc+ca)}$$

$$\geq \frac{(a^2+b^2+c^2)^2}{6(a^2+b^2+c^2)}, \quad \text{since } ab+bc+ca \leq a^2+b^2+c^2$$

$$= \frac{(a^2+b^2+c^2)^2}{6}$$

More generally, if k, l, m are constants, then

$$\frac{a^3}{ka+lb+mc} + \frac{b^3}{kb+lc+ma} + \frac{c^3}{kc+la+mb} \geq \frac{a^2+b^2+c^2}{k+l+m}$$

43. (Romania, 1999) Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be real numbers such that

$$x_1 + x_2 + \cdots + x_n \geq x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

show that

$$x_1 + x_2 + \cdots + x_n \leq \frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n}$$

By the T2 Lemma,

$$\frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n} = \frac{x_1^2}{x_1 y_1} + \frac{x_2^2}{x_2 y_2} + \cdots + \frac{x_n^2}{x_n y_n}$$

$$\geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{x_1 y_1 + x_2 y_2 + \cdots + x_n y_n}$$

$$\geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{x_1 + x_2 + \cdots + x_n}$$

$$= x_1 + x_2 + \cdots + x_n$$

Arithmetic-Geometric mean Inequality

Theorem 4 *Let* $a_1, a_2, \ldots, a_n$ *be positive real numbers. The following inequality holds*

$$(a_1 a_2 \cdots a_n)^{\frac{1}{n}} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

with equality holding if and only if $a_1 = a_2 = \cdots = a_n$.

Proof 1 Let $g = (a_1 a_2 \cdots a_n)^{1/n}$ and $b_i = a_i/g$. Then

$$b_1 b_2 \cdots b_n = 1$$

and the inequality reduces to proving the following:
If $b_1 b_2, \ldots, b_n$ are positive real numbers such that $b_1 b_2 \cdots b_n = 1$ then

$$b_1 + b_2 + \cdots + b_n \geq n$$

Assume the result for $n-1$. Without loss of generality, we can assume that b_1 is the maximum of b_i and b_n is the minimum of b_i. Thus $b_1 \geq 1$ and $b_n \leq 1$. Thus

$$(b_1 - 1)(1 - b_n) \geq 0$$

and consequently

$$b_1 + b_n \geq b_1 b_n + 1$$

Now, since

$$b_2 \times b_3 \times \cdots \times b_{n-1} \times (b_1 b_n) = 1$$

by induction hypothesis,

$$n \leq b_2 + b_3 + \cdots + b_{n-1} + (b_1 b_n + 1)$$
$$\leq b_2 + b_3 + \cdots + b_{n-1} + (b_1 + b_n)$$

completing the induction.

Proof 2 Let

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n}, \qquad G = (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$$

If $a_1 = a_2 = \cdots = a_n$, clearly $A = G$ and equality holds. Assume that not all a_i are equal. Then we need to prove that $G < A$, or equivalently, $G^n < A^n$. That is, we need

$$a_1 a_2 \cdots a_n < A^n \tag{8}$$

The inequality is established by replacing the product on the left by successively larger products, reaching A^n in fewer than n steps. Each step in the process is described by the following algorithm.

Algorithm In any product of n numbers, replace the smallest, say x and the largest, say y, by two new factors A and $x + y - A$, where A is the arithmetic mean of the n numbers. The sum of the replaced numbers is $A + x + y - A = x + y$ and hence the arithmetic mean of the new n factors is also A. Since $x + y - A = y - (A - x) < y$, the largest among the factors

reduces at each step and since $x < A$, the smallest among the factors does not decrease. The new product is larger than the previous one since

$$A(x + y - A) - xy = (A - x)(y - A) > 0$$

The repeated application of the algorithm replaces the product $a_1 a_2 \cdots a_n$ by A^n since each step of the procedure brings in at least one A among the factors. This completes the proof. Most of the induction proofs are dull. But Cauchy's proof of the famous Arithmetic Mean - Geometric Mean inequality is a classic example of how to make an induction interesting.

Given positive real numbers $a_1, a_2, \ldots, a_n$, their Arithmetic mean is defined by

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

and their Geometric mean by

$$G = (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$$

The required inequality to be proved is $G \leq A$. Of course, in the usual induction, one can start the induction for $n = 1$, where it trivially holds and assume for $n - 1$ to deduce the inequality for n. But Cauchy's proof is a charming one – consisting of a "forward" and a "backward" step: Assuming the truth for n we deduce the inequality for $2n$. This is the forward step. Then assuming the inequality for n we deduce it for $n - 1$ and this is the backward step. Starting the induction at $n = 2$, successive application of forward step proves the inequality for all powers of 2. To prove it for a general n,

find a k such that $n \le 2^k$. Since the inequality holds for 2^k, applying the backward step successively we reach the inequality for n.

To start the induction, we prove the inequality for $n = 2$. Here, we have

$$(\sqrt{a_1} - \sqrt{a_2})^2 \ge 0$$

gives

$$\frac{a_1 + a_2}{2} \ge \sqrt{a_1 a_2}$$

Assume the inequality for n. Let $a_1, a_2, \ldots, a_{2n}$ be $2n$ positive real numbers. We have

$$A_1 = \frac{a_1 + a_2 + \cdots + a_n}{n} \ge (a_1 a_2 \cdots a_n)^{\frac{1}{n}} = G_1$$

$$A_2 = \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n} \ge (a_{n+1} a_{n+2} \cdots a_{2n})^{\frac{1}{n}} = G_2$$

Now, applying the inequality for G_1, G_2, we get

$$\frac{A_1 + A_2}{2} \ge \frac{G_1 + G_2}{2} \ge \sqrt{G_1 G_2}$$

Substituting for A_1, A_2, G_1, G_2 yields

$$\frac{a_1 + a_2 + \cdots + a_{2n}}{2n} \ge (a_1 a_2 \cdots a_{2n})^{\frac{1}{2n}}$$

Now for the proof of the backward step.

Assuming that the inequality holds for n, let $a_1, a_2, \ldots, a_{n-1}$ be $n-1$ positive real numbers. Let $g = (a_1 a_2 \cdots a_{n-1})^{\frac{1}{n-1}}$, the geometric mean of the a_is. Applying the inequality for

$a_1, a_2, \ldots, a_{n-1}, g$, we get

$$a_1 + a_2 + \cdots + a_{n-1} + g \geq n(a_1 a_2 \cdots a_{n-1} g)^{\frac{1}{n}}$$
$$= n(g^{n-1} g)^{\frac{1}{n}}$$
$$= ng$$

Thus

$$a_1 + a_2 + \cdots + a_{n-1} \geq (n-1)g$$

completing the proof.

Problems

1. Let a, b, c are the lengths of sides of a triangle. Prove that
$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3$$
   Solution Letting $a = y+z, b = z+x, c = x+y$, we need to prove
$$\frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} \geq 6$$
   We have
$$\frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} = \left(\frac{y}{x} + \frac{x}{y}\right) + \left(\frac{y}{z} + \frac{z}{y}\right) + \left(\frac{z}{x} + \frac{x}{z}\right)$$
$$\geq 2 + 2 + 2$$
$$= 6$$

2. For positive real numbers a, b, c prove that
$$\frac{a}{2a+b} + \frac{b}{2b+c} + \frac{c}{2c+a} \leq 1$$

Solution

$$\frac{a}{2a+b} + \frac{b}{2b+c} + \frac{c}{2c+a} = \frac{1}{2+b/a} + \frac{1}{2+c/b} + \frac{1}{2+a/c}$$

Put $c/b = x, a/c = y, b/a = z$. Then $xyz = 1$ and we need to prove

$$\frac{1}{2+x} + \frac{1}{2+y} + \frac{1}{2+z} \leq 1$$

This is equivalent to

$$\Leftrightarrow (2+y)(2+z) + (2+z)(2+x) + (2+x)(2+y)$$
$$\leq (2+x)(2+y)(2+z)$$
$$\Leftrightarrow 12 + 4(x+y+z) + (xy+yz+zx)$$
$$\leq 8 + 4(x+y+z) + 2(xy+yz+zx) + xyz$$
$$\Leftrightarrow 4 \leq xy + yz + zx + xyz$$

Now, $xy + yz + zx \geq 3(x^2y^2z^2)^{1/3} = 3$ and $xyz = 1$. Thus the proof is complete.

3. Let a, b, c be positive real numbers. Prove:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Solution

$$\left(1 + \frac{a}{b+c}\right) + \left(1 + \frac{b}{c+a}\right) + \left(1 + \frac{c}{a+b}\right)$$
$$= (a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right)$$
$$= \frac{1}{2}\left((a+b) + (b+c) + (c+a)\right)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right)$$
$$\geq \frac{9}{2}$$

Iranian Geometry Olympiad

Iranian Geometry Olympiad is an annual academic competition for high school students. The competition has been conducted from 2014 and the competition for this year is the sixth Iranian Geometry Olympiad.

The competition is held at three levels: Elementary for 7th and 8th grade students, Intermediate for 9th and 10th grade students and Advanced for 11th and 12th grade students.

This year, the Mathematics Teachers Association (along with Mr Ravindra Sai Durbha) conducted the Advanced level exam in India. There was a very enthusiastic response from students for the Olympiad. The exam was conducted in the following centers: Delhi, Pune, Chennai, Bangalore, Hyderabad, Vijaywada, Kolkata, Guwahati, Bhubhaneswar, Vadodara, Surat, Bhilai, Rajkot, Chandigarh, Pune, Sikar, Ranchi, Nagpur, Indore and Kota.

A total of 6169 students from 53 countries participated in the Olympiad. This year, the Mathematics Teachers Association (I) conducted the Olympiad on behalf of the IGO organizers in India. We had 441 students from India participating in the Olympiad. The results for India are shown below:

Elementary Level:

M.V.Aditya	Gold
Siddharth Choppara	Silver
Sameer	Silver
Gunjan Aggarwal	Silver
Kshitij Sodani	Silver

Intermediate Level:

Mmukul Sanjeev Khedekar	Gold
Anand	Gold
Taes Padhihary	Gold
Anushka Aggarwal	Gold

Advanced Level:

Shantanu Nene	Gold
Adhavan S.V	Silver
Anish Yogesh Kulkarni	Silver
Ritam Nag	Silver
Pranjal Srivastava	Silver

ASSOCIATION ACTIVITIES

The AMTI Executive committee meeting held on 26.09.2019 at Saradha Secondary School, Gopalapuram, Chennai -86. Our Executive Chairman Prof. J. Pandurangan, presided over the meeting. The General Secretary Dr. M. Palanivasan welcomed the members.

The following Eleven (11) members were present:

Prof. J.Pandurangan,	Sri R.Athmaraman,	Sri S.R.Santhanam
Sri G. Gnanasundaram,	Sri V.Sundaramurthy,	Dr. P.Bhakatavatsulu,
Dr. P. Bakthavachulu,	Sri Dr. M. Kumaraswamy,	Sri P.V. Swaminathan
Smt. Hemalatha Thaigarajan	Sri Sadagopan Rajesh	Dr. M. Palanivasan

The General Secretary took up the points one after the other.

1. The audited accounts and budget for 2018-2019 was finalized by our auditor and submitted in the E.C. for approval. Dr. S.R. Santhanam request to check once again the expenses of Kota conference.
2. AMTI society registration renewal work was going on and it has come to the final level. The E.C. has agreed to pay the penalty amount from 1992 and requested the General Secretary Dr. M. Palanivasan to proceed further and completed the work successfully.
3. 54th Annual conference will be held at Vivekananda Kendra on 2019 December 27, 28 and 29 in Kanyakumari General Secretary Dr. M. Palanivasan and Mr. R. Athmaraman visited Kanyakumari and note down the facilities in the venue. Later on General secretary Dr. M. Palanivasan and Conference Secretary Dr. S.R. Santhanam have visited Kanyakumari and finalized the rates for boarding and lodging.

4. Status report
    1) MT editor Dr. S. Muralidharan informed through phone that next issues will be ready in time.
    2) JM:- J.M. Editor Mr. R.Athamaraman has relinquished from editorship and Dr. R. Sivaraman will be the editor for JM from next issue and it will be getting ready.
    3) NMTC:- NMTC 2019 first level exam was went on well with no remarks. Second level exam preparation is going on and NMTC Secretary Dr. Hemalatha Thaigarajan is taking care of common centres for second level exam.
    4) Popular lectures:- The present popular lecture secretary Mr. P. Ramesh has informed that he may be relived from that post. Dr. T. Kumaraswamy has been selected unanimously as the Secretary Popular Lecture. The EC requested him to organise propular lecture once in two months without fail.

    Any other matters:-
1. A workshop on Geometry will be held in our project office for Sub-Junior and Junior students on 2019 September 28, 29 and 30.
2. Dr. Kumarasamy requested to conduct a workshop for village students.

3. The E.C members unanimously decided to restructure the salary for our office staff. It requested our GS to fix the scale of pay for each one and verify the EPF details. Also it has been decided to sanction the increment of maximum 3% of the basic + DA every year and bonus will be a fixed amount based on their service (15 days salary) not by percentage.

4. In recognition of the service rended by Sri R. Athmaraman the EC unanimously designated him as a Patron of our Association and advise may be taken from him for the development of our association. This will be placed in GB for approval.

Alas! Balu Sir, is no more!

Yes, with profound grief we record the demise of one of the veterans in the realm of Mathematics, Cryptology and Sanskrit — Dr. N. Balasubramanian, one of the architects who, as a senior expert, nurtured the Association of Mathematics Teachers of India.

Mathematics was his love, Cryptology was his forte and Sanskrit was his craze. He had an amazing mastery over subjects in different disciplines.

Even as a student, he had published a research paper in Number Theory that was quoted by the American Professor Bruce Berndt in his work on Srinivasa Ramanujan's notebooks.

He served as the Editor of *Junior Mathematician,* a publication of the AMTI, and evinced great pleasure in interacting with students who show promise in problem solving. His puzzle corner was quite popular.

His lectures and comments during the annual conferences of the association were marvels; they attested to his command over several areas, which engaged and educated the audience. His interactive sessions were filled with humour and wit. Now we would be missing them.

His work at Defence Research and Development Organization (DRDO) and Joint Cipher Bureau brought laurels to him for his expertise in Coding competence.

It is an irreparable loss to AMTI. When this writer met him about a couple of months ago, he promised to write a nice book on crypto-arithmetic for the benefit of students. Unfortunately that is one promise he could not keep.

We miss you, Balu Sir.
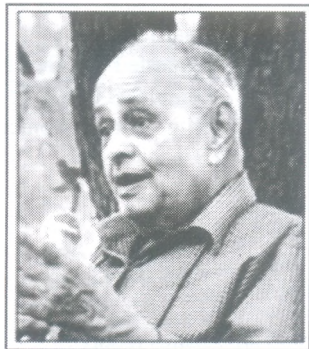
☐ *Athmaraman R.*

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